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2N-mode Squeezing Operator Gained by the Entangled State Representation

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Abstract We discuss the explicit form of the 2N-mode squeezing operator via N-pair entangled state. We also present the squeezing properties of the 2N-mode squeezed state and obtain the variances of the 2N-mode quadrature operators in 2N-mode entangled state.

Keywords 2N-mode squeezed state · Entangled state representation

1 Introduction

In recent years the entanglement has been paid much attention because their uses in quantum optics [1–3]. It was first pointed out by Einstein, Podolsky and Rosen (EPR [4] in their famous paper arguing the incompleteness of quantum mechanics. EPR introduced the wave function for two particles' relative position $Q_1 - Q_2$ (with center of mass coordinate Q_0) and their total momentum $P_1 + P_2$ (with eigenvalue P_0),

$$\Psi(Q_1, Q_2) = \frac{1}{2\pi}\int dp \exp[ip(Q_1 - Q_2 + Q_0)], \tag{1}$$

which describes a sharply correlated two-particle system. For $[Q_1 - Q_2, P_1 + P_2] = 0$, in [5, 6] and [7, 8], the simultaneous eigenstate $|\eta\rangle$ is found in two-mode Fock space,

$$|\eta\rangle = \exp\left[-\frac{1}{2}|\eta|^2 + \eta a_1^\dagger - \eta^* a_2^\dagger + a_1^\dagger a_2^\dagger\right]|00\rangle, \tag{2}$$

where $\eta = \frac{1}{\sqrt{2}}(\eta_1 + i\eta_2)$ is a complex number, $|00\rangle$ is the two-mode vacuum state, $(a_i, a_i^\dagger), i = 1, 2$ are two-mode Bose annihilation and creation operators in Fock space, re-

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lated to (Q_i, P_i) by

$$Q_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger), \qquad P_i = \frac{1}{\sqrt{2}i}(a_i - a_i^\dagger). \tag{3}$$

Based on the entangled state $|\eta\rangle$, we defined the two-mode squeezing operator as [9]

$$S_2 \equiv \int \frac{d^2\eta}{\mu\pi} \left|\frac{\eta}{\mu}\right\rangle\langle\eta|, \tag{4}$$

where $\mu = e^\lambda$ is a squeezing parameter.

With the help of the technique of integration within an ordered product (IWOP) of operators and using (4), we obtain [10–12]

$$S_2 = \exp[\lambda(a_1^\dagger a_2^\dagger - a_1 a_2)]. \tag{5}$$

For two-mode squeezing operator S_2 has been found, an interesting question thus naturally arises: Is there 2N-mode squeezing operator which squeezes the N-pair of entangled state? If yes, what is the corresponding explicit form of the 2N-mode squeezing operator?

In this paper, we shall answer these questions. The passage is arranged as follows: In Sect. 2, we discuss the explicit form of the 2N-mode squeezing operator. In Sect. 3, we present the squeezing properties of the 2N-mode squeezed state and a example is also given to prove our conclusion. In Sect. 4, the quantum mapping of $U(\Lambda)$ is given. We hope this new 2N-mode squeezing operator may have applications in quantum optical.

2 The Explicit Form of 2N-Mode Squeezing Operator

In this section, we construct a 2N-mode squeezing operator in the entangled state representation. To realize our goal, we introduce the N-pair entangled state

$$\begin{aligned} |\eta\rangle &= \left|\begin{pmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_n\end{pmatrix}\right\rangle \equiv |\eta_1, \eta_2, \ldots, \eta_n\rangle \\ &= \exp\left\{-\frac{\eta^*\widetilde{\eta}}{2} + \mathbf{A}^\dagger\widetilde{\eta} - \mathbf{B}^\dagger\widetilde{\eta}^* + \mathbf{B}^\dagger\widetilde{\mathbf{A}^\dagger}\right\}|\mathbf{00}\rangle, \end{aligned} \tag{6}$$

where $\eta_1 = \eta_{1r} + i\eta_{1c}$, $\mathbf{A}^\dagger = (a_1^\dagger, a_3^\dagger, \ldots, a_{2n-1}^\dagger)$, $\mathbf{B}^\dagger = (a_2^\dagger, a_4^\dagger, \ldots, a_{2n}^\dagger)$,

$$\widetilde{\eta} = \begin{pmatrix}\eta_1\\ \eta_2\\ \vdots\\ \eta_n\end{pmatrix},$$

and $|\mathbf{00}\rangle$ is a 2N-mode vacuum state.

Then we defined 2N-mode squeezing operator as

$$U(\Lambda) = (\det\Lambda)^{1/2} \int_{-\infty}^{\infty} \frac{d^{2n}\eta}{\pi^n} |\Lambda\eta\rangle\langle\eta|, \tag{7}$$

where Λ is a $N \times N$ real matrix.

It is easy to see

$$U^{\dagger}(\Lambda)U(\Lambda) = \mathbf{1}, \tag{8}$$

and

$$U^{\dagger}(\Lambda)Q_i U(\Lambda) = \det\Lambda \int_{-\infty}^{\infty} \frac{d^{2n}\eta}{\pi^n} \langle \Lambda\eta|\eta\rangle \eta_i \int_{-\infty}^{\infty} \frac{d^{2n}\eta}{\pi^n} |\Lambda\eta\rangle\langle\eta| = \Lambda_{ij} Q_i. \tag{9}$$

Now using the IWOP technique, we put (6) into (7) and obtain

$$\begin{aligned} U(\Lambda) &= \int_{-\infty}^{\infty} \frac{d^{2n}\eta}{\pi^n} (\det\Lambda)^{1/2} |\Lambda\eta\rangle\langle\eta| \\ &= \int_{-\infty}^{\infty} \frac{d^{2n}\eta}{\pi^n} (\det\Lambda)^{1/2} {:}\exp\left\{ -\frac{1}{2}\eta^*(\mathbf{1} + \widetilde{\Lambda}\Lambda)\widetilde{\eta} + (\widetilde{\Lambda}\mathbf{A}^{\dagger} - \mathbf{B})\widetilde{\eta} \right. \\ &\quad \left. - (\widetilde{\Lambda}\mathbf{B}^{\dagger} - \mathbf{A})\widetilde{\eta^*} + \mathbf{B}^{\dagger}\widetilde{\mathbf{A}^{\dagger}} + \mathbf{B}\widetilde{\mathbf{A}} - \mathbf{a}^{\dagger}\widetilde{\mathbf{a}} \right\}{:}. \end{aligned} \tag{10}$$

where $\mathbf{a}^{\dagger} = (a_1^{\dagger}, a_2^{\dagger}, \ldots, a_{2n}^{\dagger})$, $\mathbf{a} = (a_1, a_2, \ldots, a_{2n})$.

Using the mathematical formula [13]

$$\int \frac{d^{2n}\alpha}{\pi^n} \exp[-\alpha^* \mathbf{N}\widetilde{\alpha} + \mathbf{S}\widetilde{\alpha} + \mathbf{R}\widetilde{\alpha^*}] = [\det\mathbf{N}]^{-1} \exp[\mathbf{S}\mathbf{N}^{-1}\mathbf{R}], \tag{11}$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $\alpha^* = (\alpha_1^*, \alpha_2^*, \ldots, \alpha_n^*)$, and $\mathbf{N}$ is a $n \times n$ matrix, $\mathbf{S}, \mathbf{R}$ are n column matrices, we perform the integral in (10) and have the explicit form of $U(\Lambda)$

$$\begin{aligned} U(\Lambda) &= \frac{(\det\Lambda)^{1/2}}{\det[\frac{(\mathbf{1}+\widetilde{\Lambda}\Lambda)}{2}]} \\ &\quad \times \exp\left[-\frac{1}{2}(\widetilde{\Lambda}\mathbf{A}^{\dagger} - \mathbf{B})(\mathbf{1} + \widetilde{\Lambda}\Lambda)^{-1}(\widetilde{\Lambda}\mathbf{B}^{\dagger} - \mathbf{A}) + \mathbf{B}^{\dagger}\widetilde{\mathbf{A}^{\dagger}} + \mathbf{B}\widetilde{\mathbf{A}} - \mathbf{a}^{\dagger}\widetilde{\mathbf{a}} \right]. \end{aligned} \tag{12}$$

3 The Squeezing Properties of the 2N-mode Squeezing Operator

Theoretically, the 2N-mode squeezed state is constructed by the 2N-mode squeezing operator acting on the 2N-mode vacuum state, it follows

$$U(\Lambda)|\mathbf{00}\rangle = \frac{(\det\Lambda)^{1/2}}{\det[\frac{(\mathbf{1}+\widetilde{\Lambda}\Lambda)}{2}]} \exp\left[-\frac{1}{2}\mathbf{A}^{\dagger}\widetilde{\Lambda}(\mathbf{1} + \widetilde{\Lambda}\Lambda)^{-1}\mathbf{B}^{\dagger}\widetilde{\Lambda} \right] |\mathbf{00}\rangle \equiv ||\mathbf{00}\rangle. \tag{13}$$

For an 2N-mode system, the optical quadrature phase amplitudes can be expressed as follows:

$$X_1 = \frac{1}{2\sqrt{n}} \sum_{i=1}^{2n} Q_i, \qquad X_2 = \frac{1}{2\sqrt{n}} \sum_{i=1}^{2n} P_i, \qquad [X_1, X_2] = \frac{i}{2}, \tag{14}$$

the variances of the 2N-mode quadrature operators introduced as

$$(\Delta X_i)^2 = \langle\mathbf{00}||X_i^2||\mathbf{00}\rangle - (\langle\mathbf{00}||X_i||\mathbf{00}\rangle)^2, \quad i = 1, 2. \tag{15}$$

From (13), we have

$$\langle \mathbf{00}||X_1||\mathbf{00}\rangle = \sqrt{\frac{2}{n}}\frac{(\det\Lambda)}{[\det(\mathbf{1}+\widetilde{\Lambda}\Lambda)]^2} \tag{16}$$

$$\times \langle\mathbf{00}|\exp\left[\frac{1}{2}\mathbf{A}^\dagger\widetilde{\Lambda}(\mathbf{1}+\widetilde{\Lambda}\Lambda)^{-1}\mathbf{B}^\dagger\widetilde{\Lambda}\right]\sum_{i=1}^{2n}(a_i+a_i^\dagger)$$

$$\times \exp\left[-\frac{1}{2}\mathbf{A}^\dagger\widetilde{\Lambda}(\mathbf{1}+\widetilde{\Lambda}\Lambda)^{-1}\mathbf{B}^\dagger\widetilde{\Lambda}\right]|\mathbf{00}\rangle. \tag{17}$$

Using the generalized 'rotation' operator formula, we have

$$\begin{aligned}\exp(a_i^\dagger\Omega_{ij}a_i)a_l^\dagger\exp(-a_i^\dagger\Omega_{ij}a_i) &= a_i^\dagger(e^\Omega)_{il},\\ \exp(a_i^\dagger\Omega_{ij}a_i)a_l\exp(-a_i^\dagger\Omega_{ij}a_i) &= (e^{-\Omega})_{li}a_i,\end{aligned} \tag{18}$$

where Ω is an 2×2 invertible matrix, and Einstein's summation rule has been used, i.e. for the repeated subscripts appearing in a term, summation over 1, 2 is implied, and the identity

$$\exp(a_i^\dagger\Omega_{ij}a_j) = :\exp[a_i^\dagger(e^\Omega - I)a_j]:. \tag{19}$$

From (18), (19) can be turned into

$$\langle\mathbf{00}\|X_1\|\mathbf{00}\rangle = \langle\mathbf{00}\|\sum_{ji}[a_i^\dagger\exp\left(\mathbf{1}+\widetilde{\Lambda}\Lambda\right)_{ij}+\exp\left(-\mathbf{1}-\widetilde{\Lambda}\Lambda\right)_{ji}a_i\|\mathbf{00}\rangle = 0. \tag{20}$$

The same we can get the following relations

$$\langle\mathbf{00}\|X_2\|\mathbf{00}\rangle = 0. \tag{21}$$

Then the variances of the 2N-mode quadrature operators in 2N-mode entangled state is

$$(\Delta X_1)^2 = \langle\mathbf{00}\|X_1^2\|\mathbf{00}\rangle = \frac{1}{4n}\langle\mathbf{00}\|\left(\sum_{i=1}^{2n}Q_i\right)^2\|\mathbf{00}\rangle = \frac{1}{4n}\sum_{ji}\left(\frac{\Lambda\widetilde{\Lambda}}{2}\right)_{ji} = \frac{1}{4}e^{-2\lambda}, \tag{22}$$

and

$$(\Delta X_2)^2 = \langle\mathbf{00}\|X_2^2\|\mathbf{00}\rangle = \frac{1}{4n}\langle\mathbf{00}\|\left(\sum_{i=1}^{2n}P_i\right)^2\|\mathbf{00}\rangle = \frac{1}{4n}\sum_{ji}\left(\frac{\Lambda\widetilde{\Lambda}}{2}\right)^{-1}_{ji} = \frac{1}{4}e^{2\lambda}. \tag{23}$$

This leads to $\Delta X_1.\Delta X_2 = \frac{1}{4}$, which shows that $U(\Lambda)$ is a correct N-mode squeezing operator for the N-mode quadratures in (14) and produces the standard squeezing similar to (4).

As discussed above, we clearly see that this approach of getting the explicit form of 2N-mode squeezing operator is general. We also point out the squeezing properties of the 2N-mode squeezing operator by the 2N-mode squeezing operator acting on the 2N-mode vacuum state.

Specially, when $n = 2$, (7) reduces to

$$U(\Lambda_2) = \int\frac{d^2\eta_1 d^2\eta_2}{\pi^2}\left|\begin{pmatrix}A & B\\ C & D\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix}\right\rangle\left\langle\begin{pmatrix}\eta_1\\ \eta_2\end{pmatrix},\right| \tag{24}$$

where A, B, C and D are 2×2 complex matrices.

If we let

$$A = \begin{pmatrix} \tau & 0 \\ 0 & \mu \end{pmatrix}, \qquad B = \begin{pmatrix} 0 & -\nu \\ -\sigma & 0 \end{pmatrix},$$
$$C = \begin{pmatrix} 0 & -\nu \\ -\sigma & 0 \end{pmatrix}, \qquad D = \begin{pmatrix} \tau & 0 \\ 0 & \mu \end{pmatrix}, \tag{25}$$

where μ, ν, σ and τ are complex numbers, and satisfy the condition

$$\mu\tau - \sigma\nu = 1, \tag{26}$$

then from (12), (24) becomes

$$U(\Lambda_2) = (\mu\tau)^{-1}\colon\!\exp\left\{-\frac{\nu}{\mu}a_1^\dagger a_3^\dagger - \frac{\sigma}{\tau}a_2^\dagger a_4^\dagger + \left(\frac{1}{\mu} - 1\right)(a_1^\dagger a_1 + a_3^\dagger a_3) + \left(\frac{1}{\tau} - 1\right)(a_2^\dagger a_2 + a_4^\dagger a_4) + \frac{\sigma}{\mu}a_1 a_3 + \frac{\nu}{\tau}a_2 a_4\right\}\colon. \tag{27}$$

When we let $\mu = \tau = \cosh\theta$, $\nu = \sigma = \sinh\theta$, (27) becomes

$$U(\Lambda_2) = \exp[\theta(a_1 a_3 - a_1^\dagger a_3^\dagger)]\exp[\theta(a_2 a_4 - a_2^\dagger a_4^\dagger)]. \tag{28}$$

From (28), we can see $U(\Lambda_2)$ is just a product of two two-mode squeezing operators in (13).

4 The Quantum Mapping of $U(\Lambda)$

In the two-mode, the squeezing operator can be expressed as [7, 8]

$$S_2 = \frac{1}{\mu}\int \frac{d^2\eta}{\pi}\left|\frac{\eta}{\mu}\right\rangle\langle\eta| = \exp[\lambda(a_1^\dagger a_2^\dagger - a_1 a_2)]. \tag{29}$$

As we know Q_i and P_i are the coordinate and momentum operators related to the Bose operators $(a_i, a_i^\dagger)$ by

$$Q_i = \frac{1}{\sqrt{2}}(a_i + a_i^\dagger), \qquad P_i = \frac{1}{\sqrt{2}i}(a_i - a_i^\dagger), \tag{30}$$

(7) can be changed into

$$U(\Lambda) = \exp[i\lambda QAP], \tag{31}$$

where

$$Q = \begin{pmatrix} Q_1 & Q_2 & \cdots & Q_{2n} \end{pmatrix}, \qquad A = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}_{2n\times 2n}, \qquad P = \begin{pmatrix} P_1 \\ P_2 \\ \vdots \\ P_{2n} \end{pmatrix}. \tag{32}$$

In order to know the transformation caused by $U(\Lambda)$, we need to derive the normal ordering form of the unitary operator $U(\Lambda)$ by virtue of the IWOP technique. Using the Baker-Hausdorff formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots, \tag{33}$$

we have

$$\begin{aligned} U^\dagger(\Lambda) Q_k U(\Lambda) &= Q_k - \lambda Q_i A_{ik} - \frac{\lambda^2}{2!} Q_i A_{il} A_{lk} - \cdots = Q_i (e^{-\lambda A})_{ik}; \\ U^\dagger(\Lambda) P_k U(\Lambda) &= P_k + \lambda A_{kj} P_j + \frac{\lambda^2}{2!} A_{kl} A_{lm} P_m + \cdots = (e^{\lambda A})_{ki} P_i. \end{aligned} \tag{34}$$

So we can see $U(\Lambda)$ is the corresponding operator which leads to the classic transformation $Q_k \to Q_i(e^{-\lambda A})_{ik}$ and $P_k \to (e^{\lambda A})_{ki} P_i$.

5 Summary

In summary, we have generalized the concept of two-particle EPR entangled state to the case of 2N particle systems. With the help of IWOP technique, we construct the 2N-mode squeezing operator and the squeezing properties of it is also discussed. We also use this operator to discuss a special case which the two-pair entangled states are involved. When Λ is a special matrices, we find $U(\Lambda)$ can be a product of N two-mode squeezing operators. Its further applications in the study of multipartite teleportation, quantum dense coding and entangled fractional Fourier transformation are under consideration.

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